Edgeworth Expansions

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♦ Outline

- **Basic Idea**
- **Charlier Differential Series**
- **Edgeworth Expansions**
- **Applications**
  - Numerical Examples
  - Bootstrapping Studentized Statistics
- **Conclusions**

Try to improve CLT by approx.
$F_n(t)$ by $\sum_{j=0}^{\infty} \frac{A_j(t)}{n^{j/2}}$ such that

$$\left| F_n(t) - \sum_{j=0}^{r} \frac{A_j(t)}{n^{j/2}} \right| \leq \frac{C_r(t)}{n^{(r+1)/2}}$$

- $\lim\limits_{r\to\infty} \sum\limits_{j=0}^{r} \frac{A_j(t)}{n^{j/2}}$ may or may not exist for fixed n;

- error is of the same order of magnitude as the first neglected term

♦ Basic Idea

Can try to improve the accuracy by using e.g. Edgeworth expansions;
cf. Feller(1971), Ch. 16.
They are obtained by a Taylor expansion of the characteristic function of the statistic of interest around 0, i.e. at the center of the distribution, followed by a Fourier inversion.

This leads to expansions of the distribution in powers of $n^{-1/2}$, where the leading term is the normal density.

♦ Charlier Differential Series (1905)

Distr.	$H(x)$	$G(x)$
Charact. fct.	$\chi(u)$	$\xi(u) = \int e^{iux} dG(x)$
Cumulants	β_r	$\gamma_r = (-i)^r \frac{d^r}{du^r} \log \xi(u)\vert_{u=0}$

Let us first express $H(x)$ through $G(x)$ by means of their characteristic functions:

$$\chi(u) = \exp\{\sum_{r=1}^{\infty} (\beta_r - \gamma_r) \frac{(iu)^r}{r!}\} \xi(u)$$

Suppose all derivatives of G vanish at the extremes of the range of x. Then, by integration by parts

$(iu)^r \xi(u)$ is the Fourier transform of $(-1)^r G^{(r)}(x)$

Then, by Fourier inversion:

$$H(x) = \exp\{\sum_{r=1}^{\infty} (\beta_r - \gamma_r) \frac{(-D)^r}{r!}\} G(x)$$

D: differential operator which denotes differentiation with respect to x.

- $r = 1$:

$$
\begin{aligned}
\xi(u) &= \int_{-\infty}^{+\infty} e^{iux} g(x)dx \\
&= \underbrace{\left[\frac{1}{iu} e^{iux} g(x)\right]_{-\infty}^{+\infty}}_{=0} - \frac{1}{iu} \int_{-\infty}^{+\infty} e^{iux} g'(x)dx
\end{aligned}
$$

Thus,

$$(iu)\xi(u) = \int_{-\infty}^{+\infty} e^{iux}(-g'(x))dx$$

is the Fourier inverse of $-g'(x)$.

-

$$
\begin{aligned}
y &= a_1 x + a_2 x^2 + \dots \\
e^y &= 1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + \dots
\end{aligned}
$$

♦ Edgeworth Expansions

$$
\begin{array}{llll}
X_1, \ldots, X_n \text{ iid} & E(X_i) = 0 & = \chi_1(X_i) & \\
 & var(X_i) = \sigma^2 & = \chi_2(X_i) & \\
 & r \geq 3 & \chi_r(X_i) & = \lambda_r \sigma^r
\end{array}
$$

$$
F_n(t) = P[T_n < t] \ , \quad T_n = \frac{1}{n^{1/2}} \sum_{i=1}^{n} \frac{X_i}{\sigma}
$$

Edgeworth uses Charlier differential series to approx. $F_n(t)$ by $\Phi(t)$ (cumul. normal) expanding $\exp\{\cdots\}$ and collecting terms according to powers of $\frac{1}{n^{1/2}}$.

$$\begin{array}{ccc}
H(\cdot) = F_n(\cdot) & & G(\cdot) = \Phi(\cdot) \\
\chi(\cdot) = \psi_n(\cdot) & & \xi(u) = \exp\{-u^2/2\} \\
\beta_1 = \chi_1(T_n) = 0 & = & \gamma_1 = 0 \\
\beta_2 = \chi_2(T_n) = 1 & = & \gamma_2 = 1 \\
\beta_r = \chi_r(T_n) = \frac{\lambda_r}{n^{r/2-1}} & & \gamma_r = 0 \quad r \geq 3
\end{array}$$

$$
\begin{aligned}
\psi_n(u) &= \exp\left\{\sum_{r=3}^{\infty} \frac{\lambda_r}{n^{r/2-1}} \frac{(iu)^r}{r!}\right\} \cdot e^{-u^2/2} \\
&= \exp\left\{\frac{\lambda_3}{n^{1/2}} \frac{(iu)^3}{3!} + \frac{\lambda_4}{n} \frac{(iu)^4}{4!} + \frac{\lambda_5}{n^{3/2}} \frac{(iu)^5}{5!} \right. \\
&+ \left. \cdots \right\} \cdot e^{-u^2/2}
\end{aligned}
$$

Collecting terms of order $\frac{1}{n^{1/2}}, \frac{1}{n}, \frac{1}{n^{3/2}}, \cdots$, we get:

$$
\begin{aligned}
\psi_n(u) &= \Big\{ 1 \\
&+ \frac{1}{n^{1/2}} \lambda_3 \frac{(iu)^3}{6} \\
&+ \frac{1}{n} \left[\frac{1}{2} \lambda_3^2 \frac{(iu)^6}{(3!)^2} + \lambda_4 \frac{(iu)^4}{4!}\right] \\
&+ \frac{1}{n^{3/2}} \cdots \Big\} \cdot e^{-u^2/2}
\end{aligned}
$$

By Fourier inversion, we obtain the Edgeworth expansion for the cumulative distribution:

$$\begin{aligned} F_n(t) &= \Phi(t) \\ &- \frac{1}{n^{1/2}}\frac{\lambda_3}{6}\Phi^{(3)}(t) \\ &+ \frac{1}{n}\Big[\frac{\lambda_4}{24}\Phi^{(4)}(t) + \frac{\lambda_3^2}{72}\Phi^{(6)}(t)\Big] \\ &+ \cdots \end{aligned}$$

For the density:

$$\begin{aligned} f_n(t) &= \phi(t) \\ &- \frac{1}{n^{1/2}}\frac{\lambda_3}{6}\phi^{(3)}(t) \\ &+ \frac{1}{n}\Big[\frac{\lambda_4}{24}\phi^{(4)}(t) + \frac{\lambda_3^2}{72}\phi^{(6)}(t)\Big] \\ &+ \cdots \\ &= \phi(t)\Big\{1 \\ &+ \frac{1}{n^{1/2}}\frac{\lambda_3}{6}H_3(t) \\ &+ \frac{1}{n}\Big[\frac{\lambda_4}{24}H_4(t) + \frac{\lambda_3^2}{72}H_6(t)\Big] \\ &+ \cdots\Big\} \end{aligned}$$

where

$$\begin{aligned} H_3(t) &= t^3 - 3t \\ H_4(t) &= t^4 - 6t^2 + 3 \\ H_6(t) &= t^6 - 15t^4 + 45t^2 - 15 \end{aligned}$$

are the Hermite polynomials.

Remarks

- $\frac{1}{n^{1/2}}$ term (λ_3) corrects for skewness and $\frac{1}{n}$ term (λ_4) for kurtosis.

- In particular:

$$f_n(0) = \frac{1}{\sqrt{2\pi}}\{1+\frac{1}{n}[\frac{1}{8}\lambda_4-\frac{5}{24}\lambda_3^2]+O(n^{-2})\}$$

- If the first K moments exist, then as $n \to \infty$:

$$F_n(t)-\Phi(t)-\sum_{r=3}^{K}\frac{P_r(-\Phi(t))}{n^{r/2-1}} = o\Big(\frac{1}{n^{K/2-1}}\Big),$$

where $P_r(\cdot)$ is a polynomial of degree $3(r-2)$ with coefficients depending only on $\lambda_3, \cdots, \lambda_r$.

- The Edgeworth approximation is not a distribution function:
  $0-1$ range and monotonicity can be violated in parts of one or both tails.
  Numerical instability.

- Fisher-Cornish expansion for quantiles.

- Edgeworth expansion with respect to (e.g.) χ^2 distribution instead of the normal
  $\rightarrow$ Laguerre polynomials instead of Hermite polynomials.

Some References

Wallace(1958)
Feller(1971)
Bickel (1974), *Ann. Stat*
Albers, Bickel, van Zwet (1976), *Ann. Stat*

For $L-$ statistics:
Helmers (1979, 1980), *Ann. Stat.*

For $U-$ statistics:
Callaert, Janssen, Veraverbeke (1980), *Ann. Stat.*
Bickel, Götze, van Zwet (1986), *Ann. Stat.*

Multivariate Edgeworth expansions:
Skovgaard (1986), *Int. Stat. Rev.*

♦ Applications

Numerical Examples

See Chapter 2 in
Field, C. and Ronchetti, E. (1990)
Small Sample Asymptotics
IMS Lectures Notes

Bootstrapping Studentized Statistics

We want to compare the bootstrap distribution of a statistic $\widehat{\theta}$ with its studentized version.

Edgeworth expansions for the original statistic:

$$
\begin{aligned}
P_F[\sqrt{n}(\widehat{\theta} - \theta) \leq t] &= \Phi(\frac{t}{\sigma}) + \frac{1}{\sqrt{n}} p(\frac{t}{\sigma})\phi(\frac{t}{\sigma}) \\
&+ O(\frac{1}{n}) \qquad (1) \\
P_F[\sqrt{n}\left(\frac{\widehat{\theta} - \theta}{\widehat{\sigma}}\right) \leq t] &= \Phi(t) + \frac{1}{\sqrt{n}} q(t)\phi(t) \\
&+ O(\frac{1}{n}) \qquad (2)
\end{aligned}
$$

Edgeworth expansions for the bootstrap version:

$$
\begin{aligned}
P_{\widehat{F}}[\sqrt{n}(\widehat{\theta}^* - \widehat{\theta}) \leq t] &= \Phi(\frac{t}{\widehat{\sigma}}) + \frac{1}{\sqrt{n}} p(\frac{t}{\widehat{\sigma}})\phi(\frac{t}{\widehat{\sigma}}) \\
&+ O_p(\frac{1}{n}) \qquad (3) \\
P_{\widehat{F}}[\sqrt{n}\left(\frac{\widehat{\theta}^* - \widehat{\theta}}{\widehat{\sigma}^*}\right) \leq t] &= \Phi(t) + \frac{1}{\sqrt{n}} \widehat{q}(t)\phi(t) \\
&+ O_p(\frac{1}{n}) \qquad (4)
\end{aligned}
$$

Comparison:

$$
\begin{aligned}
(3)-(1) &= O_p\Big(\Phi(\frac{t}{\widehat{\sigma}})-\Phi(\frac{t}{\sigma})\Big)\\
&= O_p(\widehat{\sigma}-\sigma)\\
&= O_p(\frac{1}{\sqrt{n}})\\
(4)-(2) &= O_p\Big(\frac{1}{\sqrt{n}}(\widehat{q}(t)-q(t))\Big)\\
&= O_p\Big(\frac{1}{\sqrt{n}}(\widehat{\sigma}^*-\widehat{\sigma})\Big)\\
&= O_p(\frac{1}{n})
\end{aligned}
$$

Bootstrapping the studentized statistic captures the $\frac{1}{\sqrt{n}}$ term of the Edgeworth expansion and provides second order correctness.

Hall(1986, 1988)

♦ Conclusions

- Useful theoretical tool to study the higher-order properties of estimators and test statistics.

- By construction Edgeworth expansions provide in general a good approximation in the center of the density, but they can be inaccurate in the tails, where they can even become negative.

- Absolute error.